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A QUICK PRIMER ON LOGIC AND RATIONALITY

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ABSTRACT. Logic does not permit contradictions. Ordinary language sometimes uses contradictions meaningfully. Examples are provided of logical derivations. Yet one might ask how one might prove that logic itself is rational. Gottlob Frege answered that logic constitutes rationality.

Logic and mathematics have a single unshakable rule: there shall be no contradictions. Here is a contradiction you are not allowed to have: a equals not**-a,** or in symbols, **a = ~ a.**

Suppose you are in London and someone calls on your cell phone to ask whether it is raining. You might answer with perfect comprehensibility: "well, it is and it isn't." If asked whether you liked *Lord of the Rings* you might say, "I did and I didn't."[1] If a teenager says she is in a love-hate relationship with her mother, we may have a fairly good idea of what she means despite the surface contradiction.[2] Even the word *inconsistent* is often used elliptically instead of formally, as Judge Richard Posner clearly intended when he asserted in reference to the Supreme Court: "The approach of the euthanasia decisions is inconsistent with that of the abortion decisions."[3]

Here is a more difficult example of surface contradiction. When I asked members of the Jesus Seminar to cite a statement by Jesus that had no parallel or

[1] These examples derive from P.F. Strawson, Introduction to Logical Theory 7 (1952).
[2] See Leon Festinger, Theory of Cognitive Dissonance (1957).
[3] Richard A. Posner, The Problematics of Moral and Legal Theory 135 (1999).

precursor in the pre-Christian Era literature, their reply was the following statement found in all four Gospels "For whoever wants to save his life will lose it, but whoever loses his life for me will save it." Assuming it had never been said before, we are entitled to ask what it means. In its context, the statement may have been referring to two lives. Let us call "life on earth" L_1, and "life in heaven" L_2. Then we have: "Whoever wants to save L_1 will lose L_2, but whoever loses L_1 for me will save L_2." Yet we still haven't figured out a rational meaning that Jesus might have had in mind. (Surely we are entitled to assume that the Son of God was coherent and rational.) There is a context in which the quoted statement makes sense. When Christians, as followers of Jesus, were beginning to be persecuted by the Romans for their beliefs, a number of them became martyrs for Christ. Thus in the persecution context, a martyr is one who loses his L_1 for Jesus' sake and thus gains L_2. A person who refuses martyrdom is trying to save his L_1 but his act of cowardice will cost him L_2. Hence by using an appropriate context we have arrived at an intelligible meaning for the full quotation.[4]

In contrast to the foregoing examples, there are expressions that defeat themselves completely because of their logical inconsistency. If Judge Jones at the end of the trial finds that the law favors the plaintiff and yet decides the case for the defendant without further explanation, she has contradicted herself. Of course, the contradiction is implicit in the context; it can be made explicit by the addition of

[4] Yet it comes at the expense of contradicting the initial premise that Jesus said those words. For during his lifetime, when Christianity was getting started, there was no persecution; Christianity was a minuscule sect beneath Roman notice. Therefore Jesus could not have been talking about martyrdom, a phenomenon that arose a hundred years after his death. Hence there is strong evidence to believe that the words attributed to Jesus were interpolated by the authors of the Gospels or their copyists, all of whom wrote decades after Jesus' time on earth.

the suppressed premise: “a litigant wins a case if the law favors his position.” But what if Judge Jones responds that judicial discourse follows its own internal logic which sometimes differs from ordinary logic?[5] How would we deal with a response like that?

Contradiction is a term that derives its meaning from deductive logic including syllogistic and enthymematic reasoning.[6] Deductive logic is founded upon the axiom of consistency.[7] One can construct a logical system by starting with the denial of contradiction and adding some elementary connectives—logic gates in computerese—such as *not*, *and*, and *or*. We can further stipulate that a *true* statement is one that is consistent, whereas a *false* statement is one that is inconsistent. In symbols:

p = a proposition

v = or

• = and

~ = not

⊃ = if . . . then

≡ = the equivalent of

[5] Judge Posner has come close by saying that “judges are not *bound* by the rules to do anything.” Richard A. Posner, The Problems of Jurisprudence 47 (1990). However, the context of his statement was that of legal rules and not logical rules. I have no quarrel with Posner’s view of legal rules that a person is rationally free to violate a legal rule if he is aware of the cost of the sanction. There is nothing *inconsistent* about violating a legal rule; bank robbers may act illegally but not inconsistently.

[6] By the addition of a few nonambiguous symbols, or alternatively by Venn diagrams, deductive logic can entirely account for all the classic and medieval forms of syllogistic reasoning. See Willard Van Orman Quine, Methods of Logic 102-08 (4th ed. 1982).

[7] So of course is mathematics if it is taken as an analytic (and not as a synthetic) system. Carnap neatly skewered those who (like Kant) championed the synthetic view of mathematics by pointing out that it “would lead to the unacceptable consequence that an arithmetical statement might possibly be refuted tomorrow by new experiences.” Rudolf Carnap, *Intellectual Autobiography*, in Paul Arthur Schilpp (ed.), The Philosophy of Rudolf Carnap 64 (1963). Is there a possible theory of mathematics that would allow for empirical revision of (some of) its propositions? For a controversial answer grounded in the notion that all knowledge is empirical, see Saul A. Kripke, Wittgenstein on Rules and Private Language (1982).

(1) $\mathbf{p \bullet \sim p}$ **=** FALSE (a self-contradiction)

(2) $\mathbf{\sim (p \bullet \sim p)}$ **=** negation of (1) is TRUE

(3) $\mathbf{p} \lor \mathbf{\sim p}$ **=** TRUE (a tautology)

Recall the title of the motion picture comedy of 1969 about American tourists in Europe: *If It's Tuesday, This Must Be Belgium*. In symbols it is simply:

(4) $T \supset \mathbf{B}$

Most college graduates are familiar with conditional statements of this type. They recognize Tuesday as the *antecedent*, Belgium as the *consequent*, and they know that an antecedent *entails* the consequent. A different way of putting it is that the consequent *follows from* the antecedent. If we are given the antecedent, our asserting the consequent yields a *valid* argument. But even though the words *entails*, *follows from*, and *validity* are heard often in legal argument, the more subtle logical transformations are not always understood. A non-logician might say: "Suppose it's Wednesday. What does that do to your original assertion?" We reply that today's being Wednesday is consistent with our original assertion. "But if it's Wednesday, are we in Luxembourg?" Our answer is that the original assertion gave us limited information. It is possible under that assertion that we could be in Luxembourg every day except Tuesday. The question where we are on Wednesday is indeterminate if we are told that today is Wednesday; but that's because it was indeterminate under the original assertion and simply remains indeterminate under all its logical transformations. However, the original statement *can* be falsified by asserting that we are in Luxembourg on a Tuesday:

(5) $\sim$ **B • T,** falsifying (4), on the assumption that in the meantime Belgium has not annexed Luxembourg.

Can we get rid of the ‘if’ construction in the movie title and thus reduce the number of symbols that we need for formal logic? Yes, by changing the title to *Either This Is Belgium Or It’s Not Tuesday*:

(6) $T \supset \mathbf{B} = \mathbf{B} \vee \sim\mathbf{T}$

Thus we have eliminated the ‘if-then’ (or ‘horseshoe’) entailment by substituting a simpler though more awkward construction using ‘either-or.’ Moreover we have avoided the use of many of the familiar legal terms: *entail, follows from, validity, conditional statements, antecedents, consequents.* Indeed, formal logic enables us to take one more rather surprising step: we can also dispense with ‘either-or.’ We only need the two simple connectives *and* and *not*. The title would then read: *It Isn’t True That Today Is Tuesday And This Isn’t Belgium*:

(7) $T \supset \mathbf{B} = \sim (\mathbf{T} \bullet \sim \mathbf{B})$

Such a title on the marquee might not draw crowds to the theatre but it means exactly the same as the original 1969 title. An advantage of the new construction is that the elimination of ‘if’ and ‘or’ greatly simplifies the kinds of logic gates needed in computers.[8]

Sometimes skeptics of logical transformations support judicial lawmaking by claiming that deductive reasoning is an art rather than a science, and that conclusions which may appear to be logically compelled are in fact artificial[9] and subject to revision by judges who believe, like Oliver Wendell Holmes, that “the life of the law has not been logic; it has been experience.”[10] They can accuse logic of merely being a big tautology.[11] Felix Cohen rightly pointed out that logic can never establish that one case is precedent for another case because “no two cases can possibly be alike in all respects.”[12] We are familiar with dissenting opinions that seem just as logically coherent as majority opinions, and we also realize that parsing the logic of both is not likely to be a fruitful way of determining which one is correct. Therefore one might ask why judges should pay attention to the dictates of logic. How can the truth of logic itself possibly be proved?

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[8]. Simplifying the chip reduces computational error. Of course errors of this type are rare even though, contrary to popular belief, the logic gates are not on-off switches of the binary type that read 0 or 1. Instead the gates read the amplitude of electrical impulses above or below a threshold. Thus there is always a remote possibility that a tiny surge in electrical power will cause one or more gates to emit a false reading. Anyway, in practice, the ‘or’ gate is usually retained. Compare formulas (6) and (7). Although (7) eliminates the “or” switch, it adds a parenthesis, thus requiring the computer to expend the additional electronic energy of keeping track of the expression within the parenthesis.

[9] This is not the same as Lord Coke’s “artificial reason” of the law.

[10] Oliver Wendell Holmes, Jr., The Common Law 1 (1881)..

[11] Wittgenstein claimed in his first book that philosophy as a whole, including logic, is at best a tautology and hence cannot teach us anything new. *See* Ludwig Wittgenstein, Tractatus-Logico-Philosophicus 169 (1922). Of course, if logic weren’t a tautology, then it would be inconsistent, and that would presumably teach us *less* than nothing! Even though it is true that the conclusion of a valid argument is implicit in the premises and hence adds no new *fact* to the real world, it is also logically true that a radio is implicit in a radio kit and yet a radio adds a lot more to the world than does a box of parts and instructions. Every mathematical proof, if correct, is a tautology. Yet no one (especially engineers) would claim that accurate mathematical reasoning can add nothing new to the universe. *See also* the increasing recognition of “emergence” in the biological and physical sciences as a principle of phase transition that does *not* involve any detectable addition or subtraction from the system’s physical components, *e.g.*, Steven Johnson, Emergence (2001).

[12] Felix S. Cohen, *Field Theory and Judicial Logic*, 59 Yale L.J. 238, 245 (1950). Cohen’s statement seems to have been poorly articulated. Suppose case A involves 100 factors and case B involves just 80 of those factors. Then A would be an overinclusive precedent for B even though the two cases are not “alike in all respects.” This can be pictured in a Venn diagram by regarding the larger circle as A and the smaller circle that is wholly inside A as B.

What indeed can we say about Judge Jones if she were to find the law in favor of the plaintiff and, for that very reason, hold for the defendant? Not only can she say that she is not bound by any rule of logic (which is to say that logic is not rationally exportable to her judicial discourse), but she could add the more powerful point that the rules of logic cannot themselves be proven except by having recourse to rules of logic, thus begging the entire question.

Gottlob Frege in 1879 provided an answer to the charge that the validation of logic involves circular reasoning. He said that the rules of logic "neither need nor admit of proof" because they are part of what it means to be rational.[13] There is no sense in asking whether logic itself is justified, Frege claimed, because logic tells us what "justification" is. Frege's idea that logic is foundational for rational thought as well as for intelligible communication was expressed in a quite different way by Strawson when he said that a man who contradicts himself "may have succeeded in exercising his vocal chords," but has not communicated with anyone: "He utters words, but does not say anything."[14] Accordingly, we could answer Judge Jones not by arguing that she is bound in some normative or moral sense to obey the rules of logic, but rather that if she chooses to violate those rules she has simply defeated her own attempt to communicate her decision to the parties and to the public. Similarly, when a person deliberately utters an illogical string of words, our inward response is not that he's doing something immoral like lying but rather that he is wasting our time and insulting our intelligence.

Frege's point can be summarized in a simple example. If we encounter the assertion that the title *It Isn't True That Today Is Tuesday And This Isn't Belgium* says the

[13] Gottlob Frege, The Foundations of Arithmetic 4 (2d ed. 1980). "[W]hat are things independent of the reason?" he asked rhetorically. *Id*. at 36.
[14] P.F. Strawson, Introduction to Logical Theory 2 (1952)

same thing as the title *If It's Tuesday, This Must Be Belgium*, there is no need for us to resort to a logician to examine the truth or falsity of the entailment. We can figure it out for ourselves.

In other words, we are all fortunately located within the realm of rationality. Rationality, and the ordinary logic that governs the derivation of one sentence from another so as to preserve rationality, is not a discourse. It is simply the necessary prerequisite of meaningful communication.[15] Or to put it another way: logic *constitutes* rationality rather than being an *instance* of it.

[15] Is arithmetic, unlike logic, a discourse? The following is a recently discovered challenge to the most fundamental basis of arithmetic, namely, counting. There are three coins on a table; how many objects on the table can we count? There is at least three, namely, the three coins. But in n-dimensional space, we can regard coin #1 as a spacio-temporal extension of coin #2, with the result that #1 and #2 together constitute one object. Now regard all three coins as spacio-temporal extensions of each other in n-dimensions, and we have a fifth object. Hence there "are" five different objects on the table. It follows that all of arithematic and all of mathematics that employs numbers are artificial constructs. Logic, in contrast, is not subject to this kind of fundamental challenge.